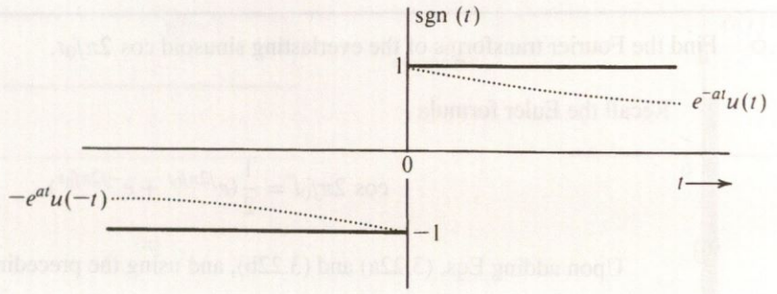

Figure 3.14
Sign function.

Therefore,

$$\begin{aligned}\mathcal{F}[\text{sgn}(t)] &= \lim_{a\to 0}\left\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right\} \\ &= \lim_{a\to 0}\left(\frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}\right) \qquad \text{(see pairs 1 and 2 in Table 3.1)} \\ &= \lim_{a\to 0}\left(\frac{-j4\pi f}{a^2+4\pi^2 f^2}\right) = \frac{1}{j\pi f} \end{aligned} \tag{3.25}$$

3.3 SOME PROPERTIES OF THE FOURIER TRANSFORM

We now study some of the important properties of the Fourier transform and their implications as well as their applications. Before embarking on this study, it is important to point out a pervasive aspect of the Fourier transform—the **time-frequency duality**.

3.3.1 Time-Frequency Duality

Equations (3.9a) and (3.9b) show an interesting fact: the direct and the inverse transform operations are remarkably similar. These operations, required to go from $g(t)$ to $G(f)$ and then from $G(f)$ to $g(t)$, are shown graphically in Fig. 3.15. The only minor difference between these two operations lies in the opposite signs used in their exponential functions.

This similarity has far-reaching consequences in the study of the Fourier transform. It is the basis of the so-called duality of time and frequency. *The duality principle may be considered by analogy to a photograph and its negative. A photograph can be obtained from its negative, and by using an identical procedure, a negative can be obtained from the photograph.* For any result or relationship between $g(t)$ and $G(f)$, there exists a dual result or relationship, obtained by interchanging the roles of $g(t)$ and $G(f)$ in the original result (along with some minor modifications arising because of a sign change). For example, the time-shifting property, to be proved later, states that if $g(t) \Longleftrightarrow G(f)$, then

$$g(t-t_0) \Longleftrightarrow G(f)e^{-j2\pi f t_0}$$

TABLE 3.1
Short Table of Fourier Transforms

	$g(t)$	$G(f)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j2\pi f}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j2\pi f}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+(2\pi f)^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j2\pi f)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j2\pi f)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$\delta(f)$	
8	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	
9	$\cos 2\pi f_0 t$	$0.5\,[\delta(f+f_0)+\delta(f-f_0)]$	
10	$\sin 2\pi f_0 t$	$j0.5\,[\delta(f+f_0)-\delta(f-f_0)]$	
11	$u(t)$	$\frac{1}{2}\delta(f)+\dfrac{1}{j2\pi f}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j2\pi f}$	
13	$\cos 2\pi f_0 t\,u(t)$	$\frac{1}{4}[\delta(f-f_0)+\delta(f+f_0)]+\dfrac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14	$\sin 2\pi f_0 t\,u(t)$	$\frac{1}{4j}[\delta(f-f_0)-\delta(f+f_0)]+\dfrac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15	$e^{-at}\sin 2\pi f_0 t\,u(t)$	$\dfrac{2\pi f_0}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
16	$e^{-at}\cos 2\pi f_0 t\,u(t)$	$\dfrac{a+j2\pi f}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
17	$\Pi\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}(\pi f\tau)$	
18	$2B\operatorname{sinc}(2\pi Bt)$	$\Pi\left(\dfrac{f}{2B}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\dfrac{\pi f\tau}{2}\right)$	
20	$B\operatorname{sinc}^2(\pi Bt)$	$\Delta\left(\dfrac{f}{2B}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$	$f_0=\dfrac{1}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-2(\sigma\pi f)^2}$	

The dual of this property (the frequency-shifting property) states that

$$g(t)e^{j2\pi f_0 t} \Longleftrightarrow G(f-f_0)$$

Observe the role reversal of time and frequency in these two equations (with the minor difference of the sign change in the exponential index). The value of this principle lies in the fact that *whenever we derive any result, we can be sure that it has a dual.* This assurance can lead to valuable insights about many unsuspected properties or results in signal processing.

The properties of the Fourier transform are useful not only in deriving the direct and the inverse transforms of many functions, but also in obtaining several valuable results in signal

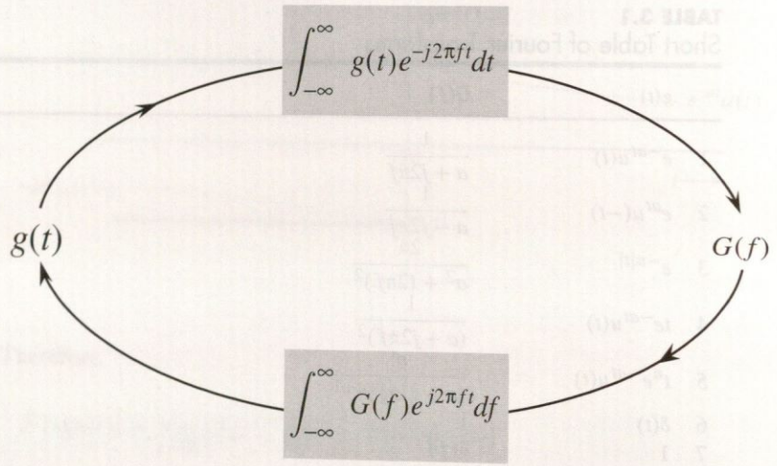

Figure 3.15 Near symmetry between direct and inverse Fourier transforms.

processing. The reader should not fail to observe the ever-present duality in this discussion. We begin with the duality property, which is one of the consequences of the duality principle discussed.

3.3.2 Duality Property

The duality property states that if

$$g(t) \Longleftrightarrow G(f)$$

then

$$G(t) \Longleftrightarrow g(-f) \tag{3.26}$$

The duality property states that if the Fourier transform of $g(t)$ is $G(f)$ then the Fourier transform of $G(t)$, with f replaced by t, is the $g(-f)$ which is the original time domain signal with t replaced by $-f$.

Proof: From Eq. (3.9b),

$$g(t) = \int_{-\infty}^{\infty} G(x)e^{j2\pi xt}\,dx$$

Hence,

$$g(-t) = \int_{-\infty}^{\infty} G(x)e^{-j2\pi xt}\,dx$$

Changing t to f yields Eq. (3.26). ■

Example 3.8 In

Figure 3.16 Duality property of the Fourier transform.

Example 3.8 In this example we shall apply the duality property [Eq. (3.26)] to the pair in Fig. 3.16a.

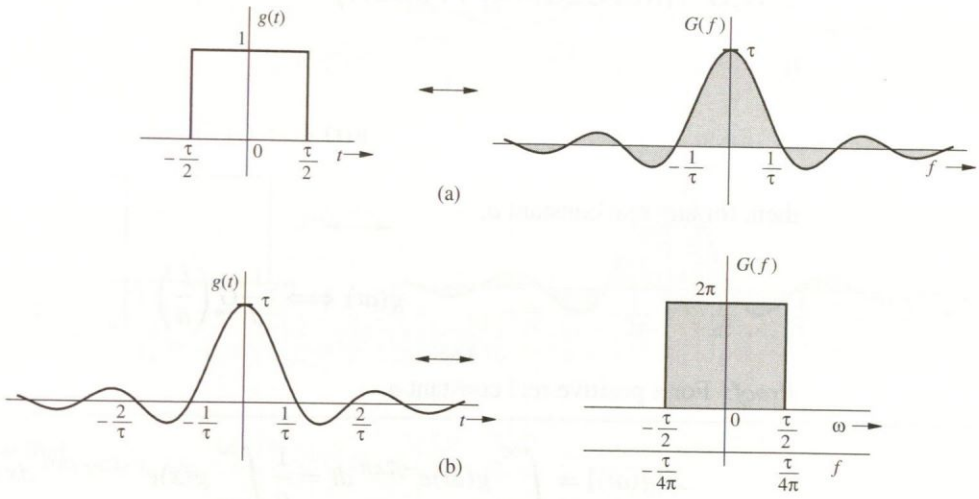

Figure 3.16 Duality property of the Fourier transform.

From Eq. (3.19) we have

$$\Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \operatorname{sinc}(\pi f \tau) \tag{3.27a}$$

$$\underbrace{\Pi\left(\frac{t}{\alpha}\right)}_{g(t)} \Longleftrightarrow \underbrace{\alpha \operatorname{sinc}(\pi f \alpha)}_{G(f)} \tag{3.27b}$$

Also $G(t)$ is the same as $G(f)$ with f replaced by t, and $g(-f)$ is the same as $g(t)$ with t replaced by $-f$. Therefore, the duality property (3.26) yields

$$\underbrace{\alpha \operatorname{sinc}(\pi \alpha t)}_{G(t)} \Longleftrightarrow \underbrace{\Pi\left(-\frac{f}{\alpha}\right)}_{g(-f)} = \Pi\left(\frac{f}{\alpha}\right) \tag{3.28a}$$

Substituting $\tau = 2\pi\alpha$, we obtain

$$\tau \operatorname{sinc}\left(\frac{\tau t}{2}\right) \Longleftrightarrow 2\pi\, \Pi\left(\frac{2\pi f}{\tau}\right) \tag{3.28b}$$

In Eq. (3.28) we used the fact that $\Pi(-t) = \Pi(t)$ because $\Pi(t)$ is an even function. Figure 3.16b shows this pair graphically. Observe the interchange of the roles of t and $2\pi f$ (with the minor adjustment of the factor 2π). This result appears as pair 18 in Table 3.1 (with $\alpha = 2B$).

As an interesting exercise, readers should apply the duality property to generate a dual of every pair in Table 3.1.

3.3.3 Time-Scaling Property

If

$$g(t) \Longleftrightarrow G(f)$$

then, for any real constant a,

$$g(at) \Longleftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right) \tag{3.29}$$

Proof: For a positive real constant a,

$$\mathcal{F}[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j2\pi ft}\, dt = \frac{1}{a} \int_{-\infty}^{\infty} g(x) e^{(-j2\pi f/a)x}\, dx = \frac{1}{a} G\left(\frac{f}{a}\right)$$

Similarly, it can be shown that if $a < 0$,

$$g(at) \Longleftrightarrow \frac{-1}{a} G\left(\frac{f}{a}\right)$$

Hence follows Eq. (3.29). ■

Significance of the Time-Scaling Property

The function $g(at)$ represents the function $g(t)$ compressed in time by a factor a ($|a| > 1$) (see Sec. 2.3.2). Similarly, a function $G(f/a)$ represents the function $G(f)$ expanded in frequency by the same factor a. *The time-scaling property states that time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression.* Intuitively, we understand that compression in time by a factor a means that the signal is varying **more rapidly** by the same factor. To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a. Similarly, a signal expanded in time varies more slowly; hence, the frequencies of its components are lowered, implying that its frequency spectrum is compressed. For instance, the signal $\cos 4\pi f_0 t$ is the same as the signal $\cos 2\pi f_0 t$ time-compressed by a factor of 2. Clearly, the spectrum of the former (impulse at $\pm 2f_0$) is an expanded version of the spectrum of the latter (impulse at $\pm f_0$). The effect of this scaling is demonstrated in Fig. 3.17.

Figure 3.17 The scaling property of the Fourier transform.

Reciprocity of Signal Duration and Its Bandwidth

The time-scaling property implies that if $g(t)$ is wider, its spectrum is narrower, and vice versa. Doubling the signal duration halves its bandwidth, and vice versa. This suggests that the bandwidth of a signal is inversely proportional to the signal duration or width (in seconds). We have already verified this fact for the rectangular pulse, where we found that the bandwidth of a gate pulse of width τ seconds is $1/\tau$ Hz. More discussion of this interesting topic can be found in the literature.[2]

Example 3.9

Figure 3.18 (a) $e^{-a|t|}$ and (b) its Fourier spectrum.

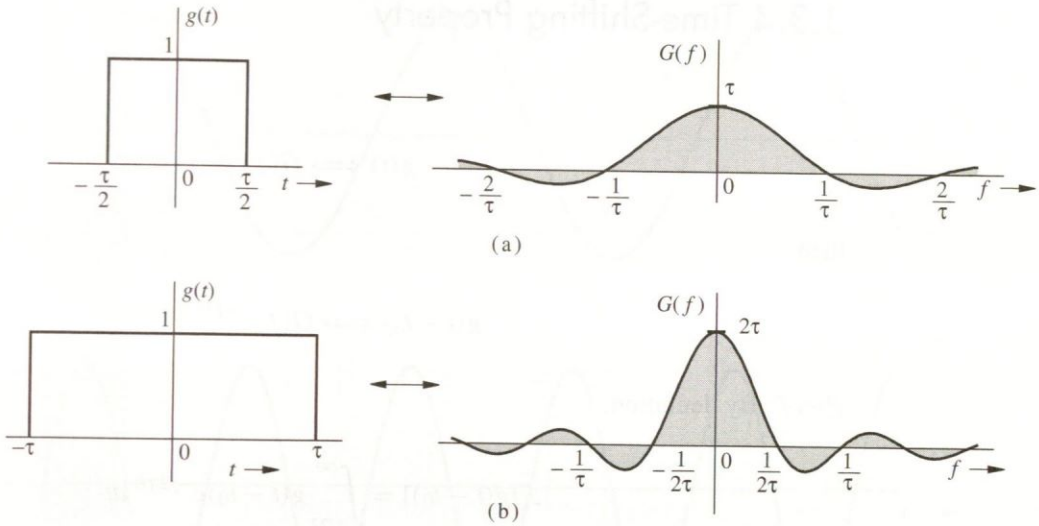

Figure 3.17 The scaling property of the Fourier transform.

Example 3.9 Show that

$$g(-t) \Longleftrightarrow G(-f) \tag{3.30}$$

Use this result and the fact that $e^{-at}u(t) \Longleftrightarrow 1/(a + j2\pi f)$ to find the Fourier transforms of $e^{at}u(-t)$ and $e^{-a|t|}$.

Equation (3.30) follows from Eq. (3.29) by letting $a = -1$. Application of Eq. (3.30) to pair 1 of Table 3.1 yields

$$e^{at}u(-t) \Longleftrightarrow \frac{1}{a - j2\pi f}$$

Also

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

Therefore,

$$e^{-a|t|} \Longleftrightarrow \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2} \tag{3.31}$$

The signal $e^{-a|t|}$ and its spectrum are shown in Fig. 3.18.

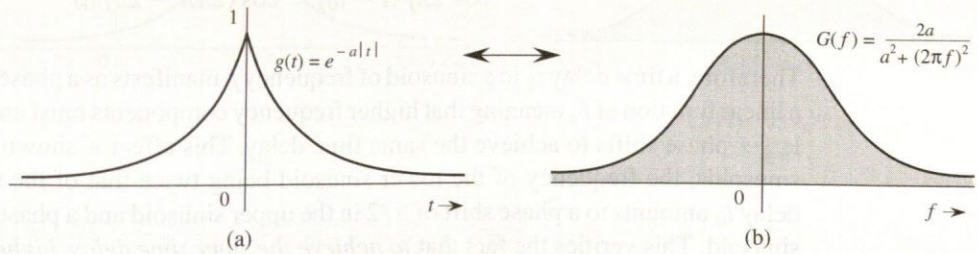

Figure 3.18 (a) $e^{-a|t|}$ and (b) its Fourier spectrum.

3.3.4 Time-Shifting Property

If

$$g(t) \Longleftrightarrow G(f)$$

then

$$g(t - t_0) \Longleftrightarrow G(f)e^{-j2\pi ft_0} \tag{3.32a}$$

Proof: By definition,

$$\mathcal{F}[g(t - t_0)] = \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft}\, dt$$

Letting $t - t_0 = x$, we have

$$\begin{aligned}\mathcal{F}[g(t - t_0)] &= \int_{-\infty}^{\infty} g(x)e^{-j2\pi f(x+t_0)}\, dx \\ &= e^{-j2\pi ft_0} \int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}\, dx = G(f)e^{-j2\pi ft_0}\end{aligned} \tag{3.32b}$$

This result shows that *delaying a signal by t_0 seconds does not change its amplitude spectrum. The phase spectrum, however, is changed by $-2\pi ft_0$.*

Physical Explanation of the Linear Phase

Time delay in a signal causes a linear phase shift in its spectrum. This result can also be derived by heuristic reasoning. Imagine $g(t)$ being synthesized by its Fourier components, which are sinusoids of certain amplitudes and phases. The delayed signal $g(t - t_0)$ can be synthesized by the same sinusoidal components, each delayed by t_0 seconds. The amplitudes of the components remain unchanged. Therefore, the amplitude spectrum of $g(t - t_0)$ is identical to that of $g(t)$. The time delay of t_0 in each sinusoid, however, does change the phase of each component. Now, a sinusoid $\cos 2\pi ft$ delayed by t_0 is given by

$$\cos 2\pi f(t - t_0) = \cos(2\pi ft - 2\pi ft_0)$$

Therefore, a time delay t_0 in a sinusoid of frequency f manifests as a phase delay of $2\pi ft_0$. This is a linear function of f, meaning that higher frequency components must undergo proportionately larger phase shifts to achieve the same time delay. This effect is shown in Fig. 3.19 with two sinusoids, the frequency of the lower sinusoid being twice that of the upper. The same time delay t_0 amounts to a phase shift of $\pi/2$ in the upper sinusoid and a phaseshift of π in the lower sinusoid. This verifies the fact that *to achieve the same time delay, higher frequency sinusoids must undergo proportionately larger phase shifts.*

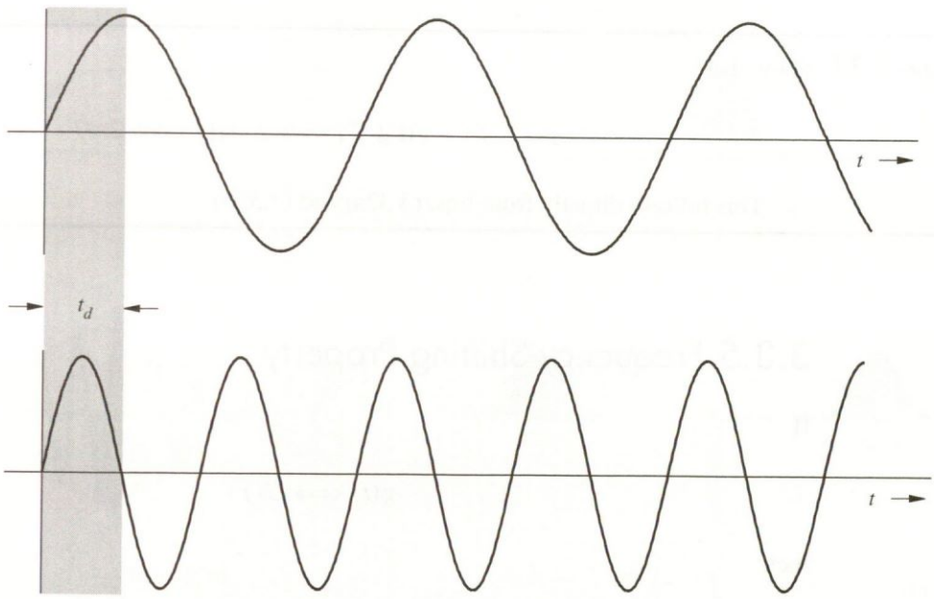

Figure 3.19 Physical explanation of the time-shifting property.

Example 3.10 Find the Fourier transform of $e^{-a|t-t_0|}$.

This function, shown in Fig. 3.20a, is a time-shifted version of $e^{-a|t|}$ (shown in Fig. 3.18a). From Eqs. (3.31) and (3.32) we have

$$e^{-a|t-t_0|} \Longleftrightarrow \frac{2a}{a^2+(2\pi f)^2} e^{-j2\pi f t_0} \tag{3.33}$$

The spectrum of $e^{-a|t-t_0|}$ (Fig. 3.20b) is the same as that of $e^{-a|t|}$ (Fig. 3.18b), except for an added phase shift of $-2\pi f t_0$.

Observe that the time delay t_0 causes a **linear** phase spectrum $-2\pi f t_0$. This example clearly demonstrates the effect of time shift.

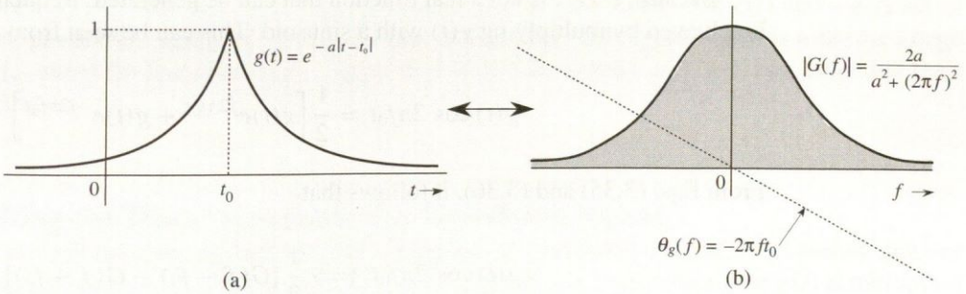

Figure 3.20 Effect of time shifting on the Fourier spectrum of a signal.

Example 3.11 Show that

$$g(t-T)+g(t+T) \Longleftrightarrow 2 \cdot G(f) \cos 2\pi fT \tag{3.34}$$

This follows directly from Eqs. (3.32a) and (3.32b).

3.3.5 Frequency-Shifting Property

If

$$g(t) \Longleftrightarrow G(f)$$

then

$$g(t)e^{j2\pi f_0 t} \Longleftrightarrow G(f-f_0) \tag{3.35}$$

This property is also called the modulation property.

Proof: By definition,

$$\mathcal{F}[g(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} g(t)e^{j2\pi f_0 t}e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} g(t)e^{-j(2\pi f-2\pi f_0)t}dt = G(f-f_0)$$

This property states that multiplication of a signal by a factor $e^{j2\pi f_0 t}$ shifts the spectrum of that signal by $f = f_0$. Note the duality between the time-shifting and the frequency-shifting properties.

Changing f_0 to $-f_0$ in Eq. (3.35) yields

$$g(t)e^{-j2\pi f_0 t} \Longleftrightarrow G(f+f_0) \tag{3.36}$$

Because $e^{j2\pi f_0 t}$ is not a real function that can be generated, frequency shifting in practice is achieved by multiplying $g(t)$ with a sinusoid. This can be seen from the fact that

$$g(t)\cos 2\pi f_0 t = \frac{1}{2}\left[g(t)e^{j2\pi f_0 t} + g(t)e^{-j2\pi f_0 t}\right]$$

From Eqs. (3.35) and (3.36), it follows that

$$g(t)\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[G(f-f_0)+G(f+f_0)] \tag{3.37}$$

This shows that the multiplication of a signal $g(t)$ by a sinusoid of frequency f_0 shifts the spectrum $G(f)$ by $\pm f_0$. Multiplication of a sinusoid $\cos 2\pi f_0 t$ by $g(t)$ amounts to modulating the sinusoid amplitude. This type of modulation is known as **amplitude modulation**. The sinusoid $\cos 2\pi f_0 t$ is called the **carrier**, the signal $g(t)$ is the **modulating signal**, and the

Figure 3.21 Amplitude modulation of a signal causes spectral shifting.

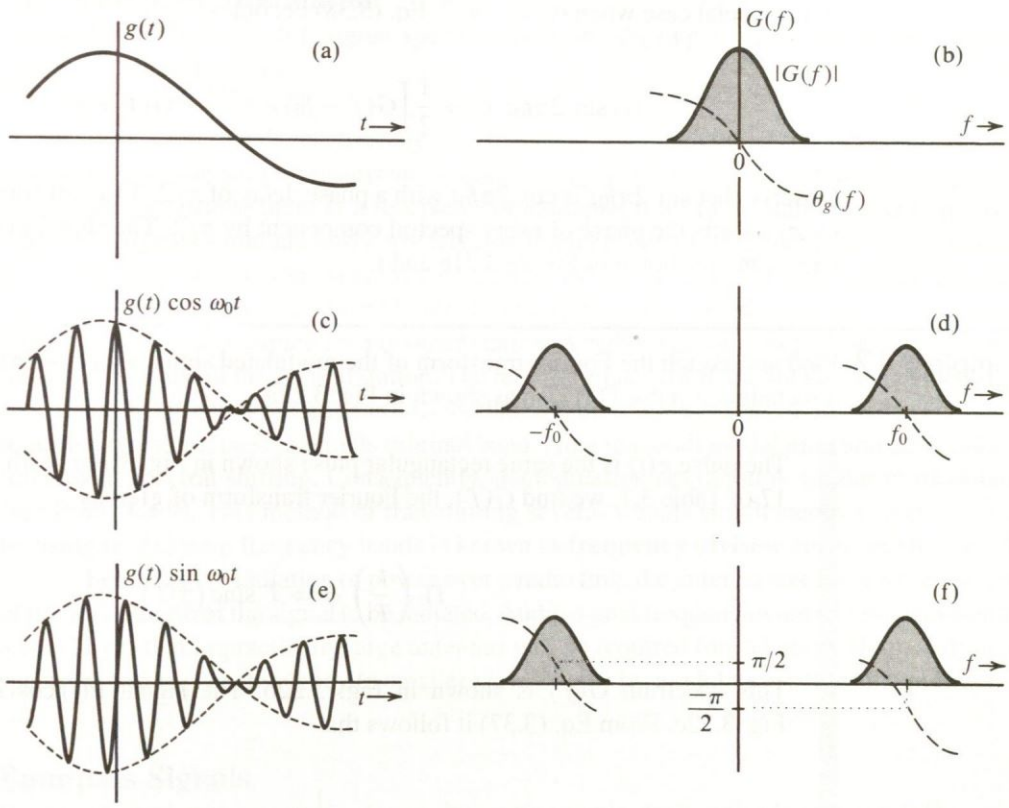

Figure 3.21 Amplitude modulation of a signal causes spectral shifting.

signal $g(t)\cos 2\pi f_0 t$ is the **modulated signal**. Modulation and demodulation will be discussed in detail in Chapters 4 and 5.

To sketch a signal $g(t)\cos 2\pi f_0 t$, we observe that

$$g(t)\cos 2\pi f_0 t = \begin{cases} g(t) & \text{when } \cos 2\pi f_0 t = 1 \\ -g(t) & \text{when } \cos 2\pi f_0 t = -1 \end{cases}$$

Therefore, $g(t)\cos 2\pi f_0 t$ touches $g(t)$ when the sinusoid $\cos 2\pi f_0 t$ is at its positive peaks and touches $-g(t)$ when $\cos 2\pi f_0 t$ is at its negative peaks. This means that $g(t)$ and $-g(t)$ act as envelopes for the signal $g(t)\cos 2\pi f_0 t$ (see Fig. 3.21c). The signal $-g(t)$ is a mirror image of $g(t)$ about the horizontal axis. Figure 3.21 shows the signals $g(t)$, $g(t)\cos 2\pi f_0 t$ and their spectra.

Shifting the Phase Spectrum of a Modulated Signal

We can shift the phase of each spectral component of a modulated signal by a constant amount θ_0 merely by using a carrier $\cos(2\pi f_0 t + \theta_0)$ instead of $\cos 2\pi f_0 t$. If a signal $g(t)$ is multiplied by $\cos(2\pi f_0 t + \theta_0)$, then using an argument similar to that used to derive Eq. (3.37), we can show that

$$g(t)\cos(2\pi f_0 t + \theta_0) \Longleftrightarrow \frac{1}{2}\left[G(f - f_0)\, e^{j\theta_0} + G(f + f_0)\, e^{-j\theta_0}\right] \tag{3.38}$$

For a special case when $\theta_0 = -\pi/2$, Eq. (3.38) becomes

$$g(t)\sin 2\pi f_0 t \Longleftrightarrow \frac{1}{2}\left[G(f-f_0)\,e^{-j\pi/2} + G(f+f_0)\,e^{j\pi/2}\right] \tag{3.39}$$

Observe that $\sin 2\pi f_0 t$ is $\cos 2\pi f_0 t$ with a phase delay of $\pi/2$. Thus, shifting the carrier phase by $\pi/2$ shifts the phase of every spectral component by $\pi/2$. The signal $g(t)\sin 2\pi f_0 t$ and its spectrum are shown in Figure 3.21e and f.

Example 3.12 Find and sketch the Fourier transform of the modulated signal $g(t)\cos 2\pi f_0 t$ in which $g(t)$ is a rectangular pulse $\Pi(t/T)$, as shown in Fig. 3.22a.

The pulse $g(t)$ is the same rectangular pulse shown in Fig. 3.10a (with $\tau = T$). From pair 17 of Table 3.1, we find $G(f)$, the Fourier transform of $g(t)$, as

$$\Pi\left(\frac{t}{T}\right) \Longleftrightarrow T\,\text{sinc}\,(\pi f T)$$

This spectrum $G(f)$ is shown in Fig. 3.22b. The signal $g(t)\cos 2\pi f_0 t$ is shown in Fig. 3.22c. From Eq. (3.37) it follows that

$$g(t)\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[G(f+f_0) + G(f-f_0)]$$

This spectrum of $g(t)\cos 2\pi f_0 t$ is obtained by shifting $G(f)$ in Fig. 3.22b to the left by f_0 and also to the right by f_0 and then multiplying them by half, as shown in Fig. 3.22d.

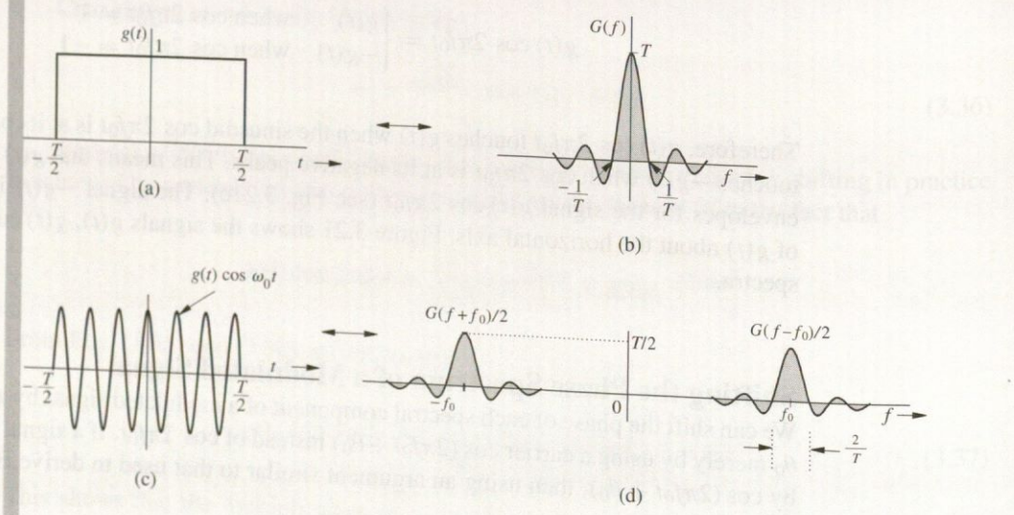

Figure 3.22 Example of spectral shifting by amplitude modulation.

Application of Modulation

Modulation is used to shift signal spectra. Spectrum shifting is necessary in the following situations (there are others).

1. If several signals, each occupying the same frequency band, are transmitted simultaneously over the same transmission medium, they will all interfere; it will be impossible to separate or retrieve them at a receiver. For example, if all radio stations decide to broadcast audio signals simultaneously, a receiver will not be able to separate them. This problem is solved by using modulation, whereby each radio station is assigned a distinct carrier frequency. Each station transmits a modulated signal, thus shifting the signal spectrum to its allocated band, which is not occupied by any other station. A radio receiver can pick up any station by tuning to the band of the desired station. The receiver must now demodulate the received signal (i.e., undo the effect of modulation). Demodulation therefore consists of another spectral shift required to restore the signal to its original band. Note that both modulation and demodulation implement spectral shifting. Consequently, demodulation operation is similar to modulation (see Prob. 3.3-9). This method of transmitting several signals simultaneously over a channel by using its different frequency bands is known as **frequency division multiplexing (FDM)**.

2. For effective radiation of power over a radio link, the antenna size must be on the order of the wavelength of the signal to be radiated. Audio signal frequencies are so low (wavelengths are so large) that impractically large antennas will be required for radiation. Here, shifting the spectrum to a higher frequency (a smaller wavelength) by modulation solves the problem.

Bandpass Signals

We have seen that if $g_c(t)$ and $g_s(t)$ are low-pass signals, each with a bandwidth B Hz or $2\pi B$ rad/s, then the signals $g_c(t)\cos 2\pi f_0 t$ and $g_s(t)\sin 2\pi f_0 t$ are both bandpass signals occupying the same band (Fig. 3.21d, f), and each having a bandwidth of $2B$ Hz. Hence, a linear combination of both these signals will also be a bandpass signal occupying the same band as that of either signal, and with the same bandwidth ($2B$ Hz). Hence, a general bandpass signal $g_{bp}(t)$ can be expressed as*

$$g_{bp}(t) = g_c(t)\cos 2\pi f_0 t + g_s(t)\sin 2\pi f_0 t \tag{3.40}$$

The spectrum of $g_{bp}(t)$ is centered at $\pm f_0$ and has a bandwidth $2B$, as shown in Fig. 3.23b. Although the magnitude spectra of both $g_c(t)\cos 2\pi f_0 t$ and $g_s(t)\sin 2\pi f_0 t$ are symmetrical about $\pm f_0$, the magnitude spectrum of their sum, $g_{bp}(t)$, is not necessarily symmetrical about $\pm f_0$. This is because the amplitudes of the two signals do not add directly with different phases for the reason that

$$a_1 e^{j\varphi_1} + a_2 e^{j\varphi_2} \neq (a_1 + a_2)e^{j(\varphi_1+\varphi_2)}$$

A typical bandpass signal $g_{bp}(t)$ and its spectra are shown in Fig. 3.23. Using a well-known trigonometric identity, Eq. (3.40) can be expressed as

$$g_{bp}(t) = E(t)\cos[2\pi f_0 t + \psi(t)] \tag{3.41}$$

* See Sec. 9.9 for a rigorous proof of this statement.

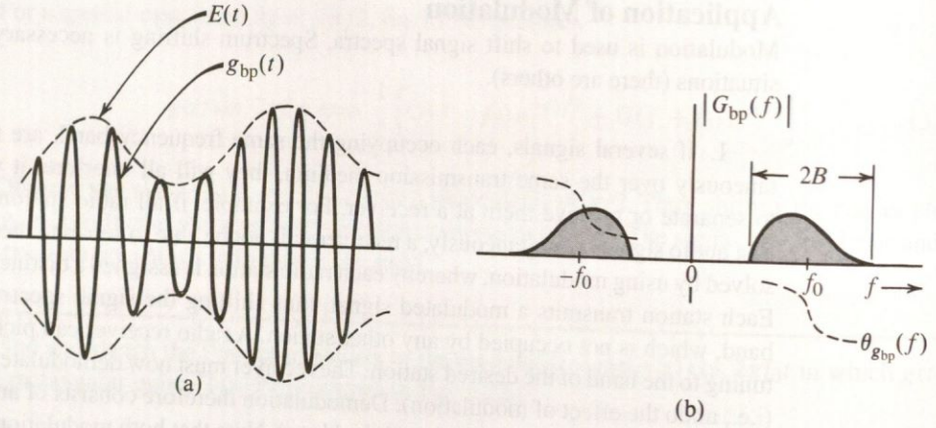

Figure 3.23 (a) Bandpass signal and (b) its spectrum.

where

$$E(t) = +\sqrt{g_c^2(t) + g_s^2(t)} \tag{3.42a}$$

$$\psi(t) = -\tan^{-1}\left[\frac{g_s(t)}{g_c(t)}\right] \tag{3.42b}$$

Because $g_c(t)$ and $g_s(t)$ are low-pass signals, $E(t)$ and $\psi(t)$ are also low-pass signals. Because $E(t)$ is nonnegative [Eq. (3.42a)], it follows from Eq. (3.41) that $E(t)$ is a slowly varying envelope and $\psi(t)$ is a slowly varying phase of the bandpass signal $g_{bp}(t)$, as shown in Fig. 3.23. Thus, the bandpass signal $g_{bp}(t)$ will appear as a sinusoid of slowly varying amplitude. Because of the time-varying phase $\psi(t)$, the frequency of the sinusoid also varies slowly* with time about the center frequency f_0.

Example 3.13 Find the Fourier transform of a general periodic signal $g(t)$ of period T_0, and hence, determine the Fourier transform of the periodic impulse train $\delta_{T_0}(t)$ shown in Fig. 3.24a.

A periodic signal $g(t)$ can be expressed as an exponential Fourier series as

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t} \qquad f_0 = \frac{1}{T_0}$$

Therefore,

$$g(t) \Longleftrightarrow \sum_{n=-\infty}^{\infty} \mathcal{F}[D_n e^{jn2\pi f_0 t}]$$

Now from Eq. (3.22a), it follows that

$$g(t) \Longleftrightarrow \sum_{n=-\infty}^{\infty} D_n \delta(f - nf_0) \tag{3.43}$$

---

* It is necessary that $B << f_0$ for a well-defined envelope. Otherwise the variations of $E(t)$ are of the same order as the carrier, and it will be difficult to separate the envelope from the carrier.

Figure 3.24 (a) Impulse train and (b) its spectrum.

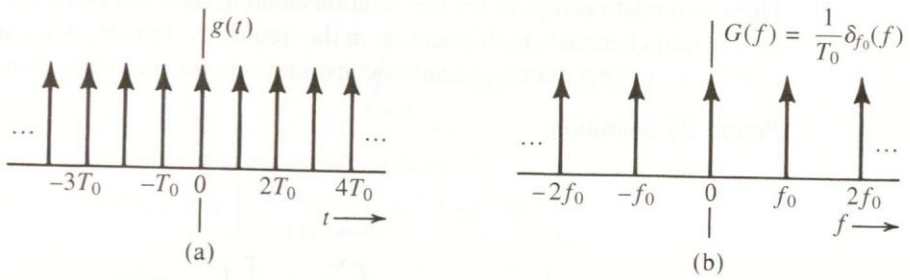

Figure 3.24
(a) Impulse train and (b) its spectrum.

Equation (2.100) shows that the impulse train $\delta_{T_0}(t)$ can be expressed as an exponential Fourier series as

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn2\pi f_0 t} \qquad f_0 = \frac{1}{T_0}$$

Here $D_n = 1/T_0$. Therefore, from Eq. (3.43),

$$\begin{aligned} \delta_{T_0}(t) &\Longleftrightarrow \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(f - nf_0) \\ &= \frac{1}{T_0} \delta_{f_0}(f) \qquad f_0 = \frac{1}{T_0} \end{aligned} \tag{3.44}$$

Thus, the spectrum of the impulse train also happens to be an impulse train (in the frequency domain), as shown in Fig. 3.24b.

3.3.6 Convolution Theorem

The convolution of two functions $g(t)$ and $w(t)$, denoted by $g(t) * w(t)$, is defined by the integral

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau) w(t - \tau)\, d\tau$$

The time convolution property and its dual, the frequency convolution property, state that if

$$g_1(t) \Longleftrightarrow G_1(f) \qquad \text{and} \qquad g_2(t) \Longleftrightarrow G_2(f)$$

then (**time convolution**)

$$g_1(t) * g_2(t) \Longleftrightarrow G_1(f) G_2(f) \tag{3.45}$$

and (**frequency convolution**)

$$g_1(t) g_2(t) \Longleftrightarrow G_1(f) * G_2(f) \tag{3.46}$$

These two relationships of the convolution theorem state that convolution of two signals in the time domain becomes multiplication in the frequency domain, whereas multiplication of two signals in the time domain becomes convolution in the frequency domain.

Proof: By definition,

$$\mathcal{F}|g_1(t) * g_2(t)| = \int_{-\infty}^{\infty} e^{-j2\pi ft}\left[\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau\right] dt$$
$$= \int_{-\infty}^{\infty} g_1(\tau)\left[\int_{-\infty}^{\infty} e^{-j2\pi ft} g_2(t-\tau)dt\right] d\tau$$

The inner integral is the Fourier transform of $g_2(t-\tau)$, given by [time-shifting property in Eq. (3.32a)] $G_2(f)e^{-j2\pi f\tau}$. Hence,

$$\mathcal{F}[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} g_1(\tau)e^{-j2\pi f\tau}G_2(f)\,d\tau$$
$$= G_2(f)\int_{-\infty}^{\infty} g_1(\tau)e^{-j2\pi f\tau}\,d\tau = G_1(f)G_2(f)$$

■

The frequency convolution property (3.46) can be proved in exactly the same way by reversing the roles of $g(t)$ and $G(f)$.

Bandwidth of the Product of Two Signals

If $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2 Hz, respectively, the bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz. This result follows from the application of the width property of convolution[3] to Eq. (3.46). This property states that the width of $x * y$ is the sum of the widths of x and y. Consequently, if the bandwidth of $g(t)$ is B Hz, then the bandwidth of $g^2(t)$ is $2B$ Hz, and the bandwidth of $g^n(t)$ is nB Hz.*

Example 3.14 Use the time convolution property to show that if

$$g(t) \Longleftrightarrow G(f)$$

then

$$\int_{-\infty}^{t} g(\tau)d\tau \Longleftrightarrow \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \tag{3.47}$$

Because

$$u(t-\tau) = \begin{cases} 1 & \tau \le t \\ 0 & \tau > t \end{cases}$$

* The width property of convolution does not hold in some pathological cases—for example, when the convolution of two functions is zero over a range, even when both functions are nonzero [e.g., $\sin 2\pi f_0 t\, u(t) * u(t)$]. Technically, however, the property holds even in this case if in calculating the width of the convolved function we take into account that range where the convolution is zero.

it follows that

$$g(t) * u(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} g(\tau)\,d\tau$$

Now from the time convolution property [Eq. (3.45)], it follows that

$$\begin{aligned} g(t) * u(t) &\Longleftrightarrow G(f)U(f) \\ &= G(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \\ &= \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \end{aligned}$$

In deriving the last result we used pair 11 of Table 3.1 and Eq. (2.18a).

3.3.7 Time Differentiation and Time Integration

If

$$g(t) \Longleftrightarrow G(f)$$

then **(time differentiation)***

$$\frac{dg(t)}{dt} \Longleftrightarrow j2\pi fG(f) \tag{3.48}$$

and **(time integration)**

$$\int_{-\infty}^{t} g(\tau)d\tau \Longleftrightarrow \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f) \tag{3.49}$$

Proof: Differentiation of both sides of Eq. (3.9b) yields

$$\frac{dg(t)}{dt} = \int_{-\infty}^{\infty} j2\pi fG(f)e^{j2\pi ft}\,df$$

This shows that

$$\frac{dg(t)}{dt} \Longleftrightarrow j2\pi fG(f)$$

Repeated application of this property yields

$$\frac{d^n g(t)}{dt^n} \Longleftrightarrow (j2\pi f)^n G(f) \tag{3.50}$$

The time integration property [Eq. (3.49)] already has been proved in Example 3.14. ■

* Valid only if the transform of $dg(t)/dt$ exists.

Example 3.15 Use the time differentiation property to, find the Fourier transform of the triangular pulse $\Delta(t/\tau)$ shown in Fig. 3.25a.

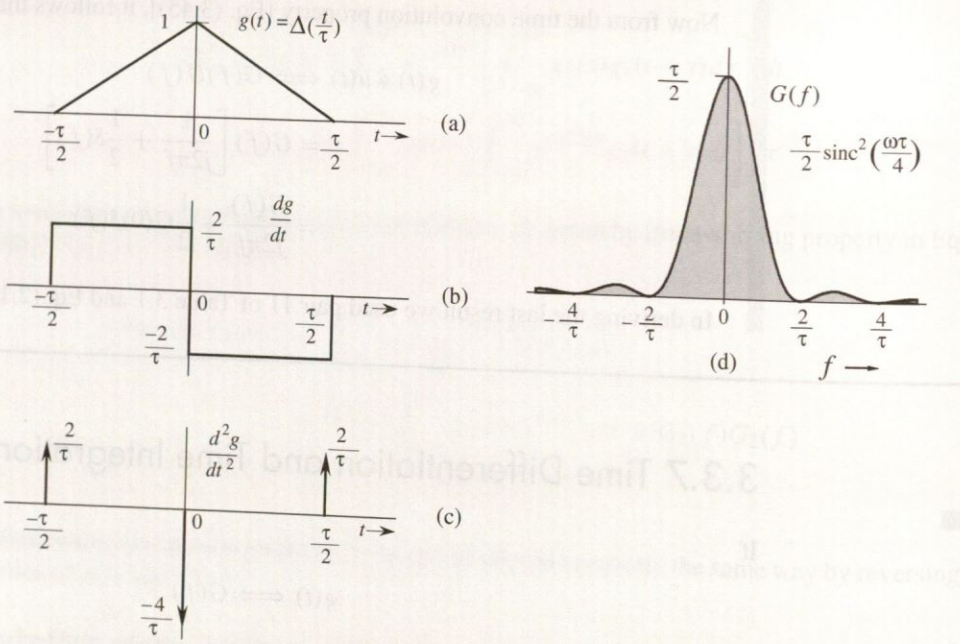

Figure 3.25 Using the time differentiation property to find the Fourier transform of a piecewise-linear signal.

To find the Fourier transform of this pulse, we differentiate it successively, as shown in Fig. 3.25b and c. The second derivative consists of a sequence of impulses, as shown in Fig. 3.25c. Recall that the derivative of a signal at a jump discontinuity is an impulse of strength equal to the amount of jump. The function $dg(t)/dt$ has a positive jump of $2/\tau$ at $t = \pm\tau/2$, and a negative jump of $4/\tau$ at $t = 0$. Therefore,

$$\frac{d^2g(t)}{dt^2} = \frac{2}{\tau}\left[\delta\left(t+\frac{\tau}{2}\right) - 2\delta(t) + \delta\left(t-\frac{\tau}{2}\right)\right] \tag{3.51}$$

From the time differentiation property (3.50),

$$\frac{d^2g}{dt^2} \Longleftrightarrow (j2\pi f)^2 G(f) = -(2\pi f)^2 G(f) \tag{3.52a}$$

Also, from the time-shifting property [Eq. (3.32a)],

$$\delta(t - t_0) \Longleftrightarrow e^{-j2\pi f t_0} \tag{3.52b}$$

Taking the Fourier transform of Eq. (3.51) and using the results in Eq. (3.52), we obtain

$$(j2\pi f)^2 G(f) = \frac{2}{\tau}\left(e^{j\pi f\tau} - 2 + e^{-j\pi f\tau}\right) = \frac{4}{\tau}\left(\cos \pi f\tau - 1\right) = -\frac{8}{\tau}\sin^2\left(\frac{\pi f\tau}{2}\right)$$

TABLE 3.2
Properties of Fourier Transform Operations

Operation	$g(t)$	$G(f)$
Superposition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Scalar multiplication	$kg(t)$	$kG(f)$
Duality	$G(t)$	$g(-f)$
Time scaling	$g(at)$	$\frac{1}{\lvert a\rvert} G\left(\frac{f}{a}\right)$
Time shifting	$g(t - t_0)$	$G(f)e^{-j2\pi f t_0}$
Frequency shifting	$g(t)e^{j2\pi f_0 t}$	$G(f - f_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Frequency convolution	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Time differentiation	$\frac{d^n g(t)}{dt^n}$	$(j2\pi f)^n G(f)$
Time integration	$\int_{-\infty}^{t} g(x)\,dx$	$\frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)$

and

$$G(f) = \frac{8}{(2\pi f)^2 \tau} \sin^2\left(\frac{\pi f \tau}{2}\right) = \frac{\tau}{2}\left[\frac{\sin(\pi f \tau/2)}{\pi f \tau/2}\right]^2 = \frac{\tau}{2}\,\mathrm{sinc}^2\left(\frac{\pi f \tau}{2}\right) \tag{3.53}$$

The spectrum $G(f)$ is shown in Fig. 3.25d. This procedure of finding the Fourier transform can be applied to any function $g(t)$ made up of straight-line segments with $g(t) \to 0$ as $|t| \to \infty$. The second derivative of such a signal yields a sequence of impulses whose Fourier transform can be found by inspection. This example suggests a numerical method of finding the Fourier transform of an arbitrary signal $g(t)$ by approximating the signal by straight-line segments.

To provide easy reference, several important properties of Fourier transform are summarized in Table 3.2.

3.4 SIGNAL TRANSMISSION THROUGH A LINEAR SYSTEM

A linear time-invariant (LTI) continuous time system can be characterized equally well in either the time domain or the frequency domain. The LTI system model, illustrated in Fig. 3.26, can often be used to characterize communication channels. In communication systems and in signal processing, we are interested only in bounded-input–bounded-output (BIBO) stable linear systems. Detailed discussions on system stability are treated in the textbook by Lathi.[3] A stable LTI system can be characterized in the time domain by its impulse response $h(t)$, which is the system response to a unit impulse input, that is,

$$y(t) = h(t) \qquad \text{when} \qquad x(t) = \delta(t)$$

The system response to a bounded input signal $x(t)$ follows the convolutional relationship

$$y(t) = h(t) * x(t) \tag{3.54}$$